Existence of solutions to principal-agent problems under general preferences and non-compact allocation space

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February 15, 2019

Abstract

We give an existence result for the principal-agent problem with adverse selection under general preferences and non-compact allocation space. The result is mainly based on the fact that the principal can always improve a feasible contract by another one which makes larger profit than the outside option from any type of agent. We also treat the case of type-dependent budget constraints.

Keywords: principal-agent problems with adverse selection, budget constraint, direct method of the calculus of variations.

1 Introduction

The principal-agent problem with adverse selection plays a distinguished role in modern microeconomic theory and has attracted a considerable amount of attention due to its numerous applications such as nonlinear pricing, taxation theory, insurance or labor contracts, just to name a few. For more on this broad topic, we refer to the classical textbook of Laffont and Tirole [8] and the references therein. It is well-known that, apart from the very special

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case of unidimensional types and quasi-linear utilities satisfying the Spence-Mirrlees single-crossing condition, these problems lead to complicated variational problems because of the non-local aspect of the incentive-compatibility condition. Even the case of a bilinear utility leads to challenging problems subject to a convexity constraint as was shown by Rochet and Choné [11]. It is therefore remarkable that Figalli, Kim and McCann [4], in the quasi-linear case and more recently, McCann and Zhang [9] for more general preferences, found conditions, connected to the regularity theory of optimal transport, under which the problem is convex and identified situations of uniqueness and stability for optimal contracts under adverse selection. However, even in these convex cases found in [4] and [9] the characterization of optimal contracts is mainly an open problem.

The situtation is even more complicated in the realistic case when agents face a type-dependent budget constraint as was considered in the works of Monteiro and Page [10] and Che and Gale [3]. And this is the very reason why we aim to revisit a simpler question, namely the existence of optimal contract issue. In the sequel, we will try to make as few structural assumptions as possible, in particular we do not assume any compactness of the allocation space or particular forms of preferences such as quasi-linearity and we also allow for type-dependent budget constrained agents. Our paper is of course very much related to the landmark paper by Monteiro and Page [10], and previous existence results of the authors, see Carlier [2] and Zhang [12]. It is also worth mentioning here the work of Ghisi and Gobbino [5] based on an original Γ-convergence approach (which, however, also assumes a compact allocation space).

Our objective is twofold. On the one hand, we wish to emphasize that compactness of the allocation space is not needed a priori, as it can indeed be guaranteed a posteriori thanks to a simple but efficient a priori estimate from Monteiro and Page [10]. This a priori estimate was in fact used a posteriori by Monteiro and Page [10] as well as by Carlier [2]. We will see, on an example in paragraph 3.3, that this trick can rather be used a priori to deal with contracts taking values in infinite-dimensional spaces. Our second aim is to present a simple strategy for proving the existence of an optimal constract when budget constraints are present which may generate some discontinuities. Monteiro and Page showed that the resulting difficulties may be overcome by a certain nonessentiality assumption, we will follow a slightly different route showing that a non-atomicity condition can be used instead.

The paper is organized as follows. Section 2 presents the model and main assumptions. In section 3, we establish an existence result. In section 4, we extend the analysis to the case of type-dependent budget constraints for the agents. Finally, we have gathered in the appendix several simple measurable selection results used throughout the paper.

2 Problem statement and assumptions

The agent's type space is a general probability space $(X, \mathcal{F}, \mu)$, the allocation space is denoted by Z and we assume that it is a Polish space (i.e. a separable and completely metrizable topological space). The agent's preferences are given by a function $U\colon X \times Z \to \mathbb{R}$ for which we assume that

$$\forall x \in X, U(x, .) \text{ is continuous on } Z, \tag{2.1}$$

and

$$\forall z \in Z,\ U(., z) \text{ is } \mathcal{F}\text{-measurable on } X. \tag{2.2}$$

Agents have access to an outside option $z_0 \in Z$. A contract is a measurable map $z : X \to Z$, and it will be called *feasible* if it is both individually-rational, i.e.,

$$U(x, z(x)) \geq U(x, z_0),\ \forall x \in X, \tag{2.3}$$

and incentive-compatible, i.e.,

$$U(x, z(x)) \geq U(x, z(x')),\ \forall (x, x') \in X \times X. \tag{2.4}$$

Finally a cost function $C : Z \to \mathbb{R} \cup \{+\infty\}$ is given for the principal which we assume to satisfy

$$C \text{ is lower semi-continuous and } C(z_0) < +\infty. \tag{2.5}$$

The principal's problem is to find a cost minimizing feasible contract that is to solve

$$\inf \Big\{ \int_X C(z(x)) \mathrm{d}\mu(x)\ : z : X \to Z \text{ feasible} \Big\}. \tag{2.6}$$

3 Existence of an optimal contract

We will prove that (2.6) admits a solution under an additional coercivity assumption. Defining

$$K := \overline{\{z \in Z\ : C(z) \leq C(z_0), \text{ and } \exists x \in X : U(x, z) \geq U(x, z_0)\}} \tag{3.1}$$

our coercivity assumption is that

$$K \text{ is compact.} \tag{3.2}$$

3.1 An a priori estimate

The main argument for the existence of a solution is based on the following observation which says that the principal can restrict herself to contracts that have smaller cost than the outside option (this argument is not new, see [10] or [2] who made a similar observation but a posteriori whereas it can be fruitfully used a priori):

Proposition 3.1. *Assume (2.1)-(2.2)-(2.5) and (3.2). Let z be a feasible contract then there exists another feasible contract $\widetilde{z}$ such that $\widetilde{z}(X) \subset K$ and*

$$\int_X C(\widetilde{z}(x))d\mu(x) \leq \int_X C(z(x))d\mu(x).$$

Proof. We may of course assume that

$$\{x \in X \ : \ C(z(x)) \leq C(z_0)\} \neq \emptyset \tag{3.3}$$

since otherwise the constant contract $\widetilde{z} \equiv z_0$ satisfies the desired claim.

Let us assume (3.3) and define for every $x \in X$,

$$u(x) := U(x, z(x)).$$

By individual-rationality and incentive-compatibility, one can write

$$u(x) = \max_{z' \in \mathcal{A}} U(x, z') \text{ where } \mathcal{A} := \{z_0\} \cup \overline{\{z(x'), \ x' \in X\}}.$$

Let us note that $\mathcal{A} \cap K \neq \emptyset$ and define

$$\widetilde{u}(x) := \max_{z' \in \mathcal{A} \cap K} U(x, z').$$

We thus have $U(x, z_0) \leq \widetilde{u}(x) \leq u(x)$ and $\widetilde{u}(x) = U(x, z(x)) = u(x)$ whenever $C(z(x)) \leq C(z_0)$. Since $\mathcal{A} \cap K$ is compact and by (2.1), for every $x \in X$ the set

$$\Gamma(x) := \{z \in \mathcal{A} \cap K \ : \ \widetilde{u}(x) = U(x, z)\}$$

is nonempty and closed. Moreover, thanks to (2.2), the set valued map Γ has an $\mathcal{F}$-measurable selection (see the Appendix for details) that we denote by $\widetilde{z}$. Since when $C(z(x)) \leq C(z_0)$, $z(x) \in \Gamma(x)$ we may also assume that $z(x) = \widetilde{z}(x)$ for every $x \in X$ for which $C(z(x)) \leq C(z_0)$. By construction $\widetilde{z}$ is individually-rational and also for every $(x, x') \in X \times X$ since $\widetilde{z}(x') \in \mathcal{A} \cap K$ we have $\widetilde{u}(x) = U(x, \widetilde{z}(x)) \geq U(x, \widetilde{z}(x'))$ so that $\widetilde{z}$ is also incentive compatible. Finally $C(\widetilde{z}(x)) = C(z(x))$ when $C(z(x)) \leq C(z_0)$ and $C(\widetilde{z}(x)) \leq C(z_0) \leq C(z(x))$ otherwise which shows that the feasible constract $\widetilde{z}$ has lower cost than the original one z and it takes by construction its values in K.

□

3.2 An existence result

Since the constant contract equal to the oustide option z_0 is feasible, the minimization problem (2.6) can be restricted to the set of feasible contracts for which

$$\int_X C(z(x))\mathrm{d}\mu(x) \le C(z_0)$$

hence for which (3.3) holds. Proposition 3.1 therefore enables us to reduce the principal's problem to the compact allocation space K (given by (3.1)) instead of Z. From this reduction, classical arguments in the lines of those of [10], [2], [12] give the existence of an optimal contract:

Theorem 3.2. *Under assumptions (2.1)-(2.2)-(2.5) and (3.2), the principal's problem (2.6) admits at least a solution.*

Proof. Let $(z_n)_n$ be a minimizing sequence for (2.6), i.e., a sequence of feasible contracts such that

$$\lim_n \int_X C(z_n(x))d\mu(x) = \inf(2.6). \tag{3.4}$$

Assuming that $\int_X C(z_n(x))d\mu(x) \le C(z_0)$ and using Proposition 3.1, we may further assume that $z_n(X) \subset K$ for each n. Then define

$$u_n(x) := U(x, z_n(x)) = \max_{z\in\mathcal{A}_n} U(x,z) \text{ where } \mathcal{A}_n := \{z_0\} \cup \overline{\{z_n(x),\ x \in X\}}.$$

Since the nonempty compact set $\mathcal{A}_n$ is included in K for every n, we may assume, taking a subsequence if necessary, that $\mathcal{A}_n$ converges to some nonempty compact subset $\mathcal{A}^*$ of K in the Hausdorff distance[1], i.e.,

$$\lim_n d_H(\mathcal{A}_n, \mathcal{A}^*) = 0. \tag{3.5}$$

Then define

$$u^*(x) := \sup_{z\in\mathcal{A}^*} U(x,z). \tag{3.6}$$

Define also the set-valued map $x \in X \mapsto \Gamma^*(x)$ by

$$\Gamma^*(x) := \{z \in K \ : \ \exists n_j \to \infty \ \text{ s.t. } \ z_{n_j}(x) \to z,\ C(z_{n_j}(x)) \to \liminf_n C(z_n(x))\}. \tag{3.7}$$

[1] Denoting by d a distance that completely metrizes the topology of the separable space Z, and by $\mathrm{dist}(A,z) := \inf_{z'\in A} d(z',z)$ the distance from z to the set A, the Hausdorff distance between the sets A and B is $d_H(A,B) := \max(\sup_{b\in B}\mathrm{dist}(A,b), \sup_{a\in A}\mathrm{dist}(B,a))$.

$\Gamma^*(x)$ is a nonempty compact set for every x and our assumptions guarantee that Γ^* has an $\mathcal{F}$-measurable selection which we denote by z^* (see Lemma 4.4 in the Appendix for details). Since $z_0 \in \mathcal{A}^*$, we have $U(x, z^*(x)) \geq U(x, z_0)$ for each $x \in X$ so that z^* is individually-rational. Let $x \in X$ and z be a cluster point of $z_n(x)$ such that $\limsup_n u_n(x) = \limsup_n U(x, z_n(x)) = U(x, z)$. It follows from (3.5) that $z \in \mathcal{A}^*$ hence

$$\limsup_n u_n(x) \leq u^*(x). \tag{3.8}$$

Now let $z \in \mathcal{A}^*$, again by (3.5), there exists a sequence $(z'_n)_n$ converging to z such that $z'_n \in \mathcal{A}_n$ for each n. By incentive-compatibility we have $u_n(x) = U(x, z_n(x)) \geq U(x, z'_n)$ for each n so that $\liminf_n u_n(x) \geq U(x, z)$. And taking the supremum in $z \in \mathcal{A}^*$ we get

$$\liminf_n u_n(x) \geq u^*(x). \tag{3.9}$$

From (3.8)-(3.9), we deduce that $u_n(x) = U(x, z_n(x))$ converges to $u^*(x)$ for each $x \in X$. Choosing a subsequence of $z_n(x)$ that converges to $z^*(x)$ therefore gives $u^*(x) = U(x, z^*(x))$. Then for any $(x', x) \in X \times X$, since $z^*(x') \in \mathcal{A}^*$ we have $u^*(x) = U(x, z^*(x)) \geq U(x, z^*(x'))$ which shows that z^* is incentive-compatible. Finally, Fatou's lemma (note that $C(z_n)$ is bounded from below by the minimum of C on the compact set K) and the fact that $z^*(x) \in \Gamma^*(x)$ where $\Gamma^*(x)$ is given by (3.7) give

$$\inf(2.6) \geq \int_X \liminf_n C(z_n(x)) d\mu(x) = \int_X C(z^*(x)) d\mu(x)$$

so that z^* solves (2.6).

□

3.3 Examples

Finite-dimensional allocations: a quasilinear case

The simplest example is the case when $z = (p, q) \in \mathbb{R} \times \mathbb{R}^d$ where $p \in \mathbb{R}$ represents the price of the contract and $q \in \mathbb{R}^d$ is the product attributes. Assume that preferences are quasi linear, i.e.,

$$U(x, z) = \varphi(x) \cdot q - p,$$

where φ is a certain measurable and bounded map from the type space X to $\mathbb{R}^d$. Let us also assume separability of the cost:

$$C(z) = c(q) - p.$$

Denoting by (p_0, q_0) the outside option, the closed set K defined in (3.1) is included in the set of $(p, q) \in \mathbb{R} \times \mathbb{R}^d$ for which

$$\sup_{x \in X} \|\varphi(x)\| \, \|q - q_0\| + p_0 \geq p, \text{ and } c(q) - p \leq c(q_0) - p_0. \tag{3.10}$$

So if the cost c is superlinear, i.e.,

$$\lim_{\|q\| \to \infty} \frac{c(q)}{\|q\|} = +\infty,$$

the bounds in (3.10) give a bound on q and then on p which implies that our assumption (3.2) is satisfied.

Finite-dimensional allocations: a non-quasilinear case

For each agent $x \in X$, the utility of purchasing a product $q \in \mathbb{R}^d$ at price $p \in \mathbb{R}$ is assumed to be

$$U(x, z) = b(x, q) - f(x, p),$$

where $z = (p, q) \in \mathbb{R} \times \mathbb{R}^d$ is the contract. For each contract z, the cost to the principal is

$$C(z) = c(q) - p.$$

Assume $b(x, \cdot)$ is Lipschitz, uniformly in x, with Lipschitz constant Lip_b, $f(x, \cdot)$ is strictly increasing in p and $\partial_p f(x, \cdot) \geq \lambda > 0$ for all x. Finally also assume that the cost c is superlinear in $q \in \mathbb{R}^d$. Then K defined in (3.1) is contained in the set of $(p, q) \in \mathbb{R} \times \mathbb{R}^d$ such that

$$c(q) \leq c(q_0) + \frac{\mathrm{Lip}_b}{\lambda} \cdot \|q - q_0\| \quad \text{and} \quad c(q) - c(q_0) + p_0 \leq p \leq p_0 + \frac{\mathrm{Lip}_b}{\lambda} \|q - q_0\|.$$

Since c is superlinear, the two conditions above imply that both p and q are bounded. Thus, K is compact.

Finite-dimensional allocations: a fully nonlinear case

Following McCann and Zhang [9], consider now a general nonlinear utility function of the form

$$U(x, z) = G(x, q, p),$$

where $z = (p, q) \in \mathbb{R} \times \mathbb{R}^d$ represents the contract with agent x. Assume G is strictly decreasing on price p, which means that the same product with a higher price provides less utility to agents.

Each contract z has a cost to the principal, which is

$$C(z) = c(q) - p.$$

Assume c is superlinear in q, G satisfies $\partial_p G(x,q,p) \leq -\lambda < 0$ for all $(x,z) \in X \times Z$ and $G(x,\cdot,p_0)$ is Lipschitz, uniformly in x, with Lipschitz constant Lip_G. To show that K is bounded, it is useful to define $K_1 = K \cap \{(q,p) \in \mathbb{R}^d \times \mathbb{R} : p \leq p_0\}$ and $K_2 = K \setminus K_1$.

By definition, we know for any $(q,p) \in K_1$,

$$c(q) - p_0 \leq c(q) - p \leq c(q_0) - p_0.$$

Since c is superlinear, this implies q is bounded. Since $c(q) - c(q_0) + p_0 \leq p \leq p_0$, p is also bounded. And thus, K_1 is bounded.

Now, if $(p,q) \in K_2$ there exists $x \in X$ such that $G(x,q,p) \geq G(x,q_0,p_0)$, but since $p > p_0$ and $\partial_p G \leq -\lambda$, using the Lipschitz assumption on $G(x,.,p_0)$, we have

$$\begin{aligned} G(x,q_0,p_0) \leq G(x,q,p) &\leq G(x,q,p_0) - \lambda(p-p_0) \\ &\leq G(x,q_0,p_0) - \lambda(p-p_0) + \mathrm{Lip}_G \|q-q_0\| \end{aligned}$$

hence

$$0 \leq p - p_0 \leq \frac{\mathrm{Lip}_G}{\lambda}\|q-q_0\|, \ c(q) \leq c(q_0) + \frac{\mathrm{Lip}_G}{\lambda}\|q-q_0\|.$$

Since c is superlinear, this implies q is bounded and p as well so K_2 is bounded. Therefore, K is compact.

Infinite-dimensional allocations

We now consider the possibility that the allocation z is infinite-dimensional, one can think for instance of a time-dependent function. We consider contracts of the form $z = (p,q)$ with $p \in \mathbb{R}$ and $q \in Z := L^2((0,T),\mathbb{R}^d)$, a utility of the form

$$U(x,z) := \int_0^T v(t,x,q(t))\mathrm{d}t - p$$

and a cost

$$C(z) = \int_0^T (c(t,q(t)) + |\dot{q}(t)|^2)\mathrm{d}t - p$$

(with $C = +\infty$ whenever $\dot{q}$ is not L^2) and an outside option (z_0, q_0) with $\dot{q}_0 \in L^2$. Then if $c(t,.)$ is superlinear uniformly in t and $v(t,x,.)$ is Lipschitz uniformly in (t,x), the set K consists of $(p,q) \in \mathbb{R} \times L^2$ such that both $\int_0^T (|q| + |\dot{q}|^2)\mathrm{d}t$ and p are uniformly bounded, it is therefore compact in $\mathbb{R} \times L^2$ by the Rellich-Kondrachov Theorem (see [1]).

4 The budget-constrained case

We now extend the model of section 2 and our existence result to the case where agents have a (type-dependent) budget constraint. This case is relevant in applications and was considered by Che and Gale [3] and was analyzed from the existence perspective by Monteiro and Page [10] who were able to deal with the discontinuity resulting from the budget constraint thanks to a specific assumption called nonessentiality that we will not use here, instead we will use a non-atomicity assumption on the type distribution.

4.1 Model and assumptions

We consider the following setting for the budget-constrained principal-agent problem. The type of the agents will consist of a preference parameter x and a budget y. The set of preference parameters is denoted by X which is equipped with a σ-algebra $\mathcal{F}$, the set of budgets is a closed interval Y with a finite lower bound $\underline{y}$ and it is equipped with its Borel algebra which we denote by $\mathcal{B}$. Contracts consist of pairs (p, q) where $p \in \mathbb{R}$ denotes the price of the contract and q denotes an allocation, while the set of allocations is denoted by Q which is assumed to be a Polish space. The outside option $(p_0, q_0) \in \mathbb{R} \times Q$ is assumed to satisfy

$$p_0 \leq \underline{y} \tag{4.1}$$

which makes it affordable even to agents with the lowest budget. Preferences are given by a function $V : X \times \mathbb{R} \times Q \to \mathbb{R}$ and we assume that

$$\forall x \in X, V(x, ., .) \text{ is continuous on } \mathbb{R} \times Q, \tag{4.2}$$

and

$$\forall (p, q) \in \mathbb{R} \times Q,\ V(., p, q) \text{ is } \mathcal{F}\text{-measurable on } X. \tag{4.3}$$

The joint distribution of types (x, y) is given by a probability measure θ on $X \times Y$ (equipped with the product σ-algebra $\mathcal{F} \otimes \mathcal{B}$). Finally the cost for the principal is given by a function $C : \mathbb{R} \times Q \to \mathbb{R} \cup \{+\infty\}$ which we assume to satisfy

$$C \text{ is lower semi-continuous and } C(p_0, q_0) < +\infty. \tag{4.4}$$

Definition 4.1. *A budget-constrained-feasible contract is an $\mathcal{F}\otimes\mathcal{B}$-measurable map $(x, y) \in X \times Y \mapsto (p(x, y), q(x, y)) \in \mathbb{R} \times Q$ that satisfies:*

- *the budget constraint: $p(x, y) \leq y$, for every $(x, y) \in X \times Y$;*

- *individual rationality,* $V(x, p(x,y), q(x,y)) \geq V(x, p_0, q_0)$*, for every* $(x,y) \in X \times Y$*;*
- *budget-constrained incentive-compatibility, i.e., for every* $(x, y, x', y') \in (X \times Y)^2$ *if* $p(x', y') \leq y$ *then*

$$V(x, p(x,y), q(x,y)) \geq V(x, p(x', y'), q(x', y')). \tag{4.5}$$

The budget-constrained principal's problem then reads

$$\inf \Big\{ \int_{X \times Y} C(p(x,y), q(x,y)) \mathrm{d}\theta(x,y) \ : (p,q) \text{ budget-constrained-feasible} \Big\}. \tag{4.6}$$

To prove that (4.6) admits solutions we shall need two more technical assumptions. The first one is a coercicity assumption similar to (3.2). Define Γ as the closure of the set of $(p,q) \in \mathbb{R} \times Q$ such that $C(p,q) \leq C(p_0, q_0)$ and there exists $(x,y) \in X \times Y$ such that $p \leq y$ and $V(x,p,q) \geq V(x, p_0, q_0)$, then our coercivity assumption is that

$$\Gamma \text{ is compact.} \tag{4.7}$$

Our last assumption is a non-atomicity condition that will enable us to deal with the possible discontinuities caused by the budget constraint on the indirect utility function. Our non-atomicity condition is that for every measurable subset S of $X \times Y$, one has[2]

$$\theta(S) = 0 \text{ whenever } S_x \text{ is at most countable for every } x \in Q. \tag{4.8}$$

Here, given $x \in X$, S_x denotes the slice $S_x := \{y \in Y \ : \ (x,y) \in S\}$.

4.2 Existence

Our first step in the existence proof is the following variant of Proposition 3.1:

Lemma 4.2. *Assume (4.1)-(4.2)-(4.3)-(4.4)-(4.7). Let* (p,q) *be a budget-constrained-feasible contract then there exists another budget-constrained-feasible contract* $(\widetilde{p}, \widetilde{q})$ *such that* $(\widetilde{p}, \widetilde{q})(X \times Y) \subset \Gamma$ *and*

$$\int_{X \times Y} C(\widetilde{p}(x,y), \widetilde{q}(x,y)) d\theta(x,y) \leq \int_{X \times Y} C(p(x,y), q(x,y)) d\theta(x,y).$$

[2]When X is a Polish space, by the disintegration Theorem, θ can be disintegrated with respect to its first marginal α as $\theta(\mathrm{d}x, \mathrm{d}y) = \theta(\mathrm{d}y|x)\alpha(\mathrm{d}x)$, in this case condition (4.8) amounts to saying that for α-a.e. x, the conditional probability $\theta(.|x)$ is atomless.

Proof. As in the proof of Proposition 3.1, there is no loss of generality in assuming that

$$\{(x,y) \in X \times Y \ : \ C(p(x,y), q(x,y)) \leq C(p_0, q_0)\} \neq \emptyset. \tag{4.9}$$

Let us define

$$v(x,y) := V(x, p(x,y), q(x,y)), \ \forall (x,y) \in X \times Y$$

and observe that by individual-rationality and budget-constrained incentive-compatilbility v can be expressed as

$$v(x,y) := \max_{(p,q) \in \mathcal{A}, \ p \leq y} V(x,p,q),$$

where

$$\mathcal{A} := \{(p_0, q_0)\} \cup \overline{\{(p(x',y'), q(x',y')), \ (x',y') \in X \times Y\}}.$$

Since $\mathcal{A} \cap \{(p,q) \in \mathbb{R} \times Q : p \leq y\} \cap \Gamma$ is non-empty and compact and thanks to (4.2), we can define the following function (that is everywhere finite):

$$\widetilde{v}(x,y) := \max_{(p,q) \in \mathcal{A} \cap \Gamma, \ p \leq y} V(x,p,q).$$

Moreover, thanks to Lemma 4.6, we can chose a maximizer $(\widetilde{p}(x,y), \widetilde{q}(x,y))$ in the program above which depends in a measurable way on (x,y) and we can also assume that

$$(\widetilde{p}(x,y), \widetilde{q}(x,y)) = (p(x,y), q(x,y)) \text{ whenever } C(p(x,y), q(x,y)) \leq C(p_0, q_0).$$

Arguing as in the proof of Proposition 3.1, we deduce that $(\widetilde{p}, \widetilde{q})$ is a budget-constrained-feasible contract and $C(\widetilde{p}(x,y), \widetilde{q}(x,y)) \leq C(p(x,y), q(x,y))$ for every $(x,y) \in X \times Y$. □

We then have the existence result:

Theorem 4.3. *Assume (4.1)-(4.2)-(4.3)-(4.4)-(4.7) and (4.8) then (4.6) admits at least a solution.*

Proof. Let (p_n, q_n) be a minimizing sequence for (4.6), thanks to Lemma 4.2 there is no loss of generality in assuming that $(p_n, q_n)(X \times Y) \subset \Gamma$ where Γ is the compact set defined above assumption (4.7). Defining

$$v_n(x,y) := V(x, p_n(x,y), q_n(x,y)), \ \forall (x,y) \in X \times Y,$$

budget-constrained feasibility then gives the representation

$$v_n(x,y) = \max_{(p,q)\in\mathcal{A}_n,\, p\le y} V(x,p,q),$$

where

$$\mathcal{A}_n := \{(p_0,q_0)\} \cup \overline{\{(p_n(x',y'), q_n(x',y')),\ (x',y')\in X\times Y\}}.$$

Since each compact set $\mathcal{A}_n$ is included in the compact set Γ, we may also assume, passing to a subsequence if necessary, that there is a compact subset $\mathcal{A}^*$ of Γ, containing (p_0,q_0) such that

$$\lim_n d_H(\mathcal{A}_n, \mathcal{A}^*) = 0. \tag{4.10}$$

Then define

$$v^*(x,y) = \max_{(p,q)\in\mathcal{A}^*,\, p\le y} V(x,p,q).$$

Thanks to Lemma 4.4, there exists a measurable selection (p^*,q^*) of the set-valued map defined for every $(x,y)\in X\times Y$ by

$$\begin{aligned}\Gamma^*(x,y) := \{(p,q)\in\Gamma\ :\ \exists n_j\to\infty\ \text{ s.t. } (p_{n_j}(x,y), q_{n_j}(x,y)) \to (p,q),\\ C(p_{n_j}(x,y), q_{n_j}(x,y)) \to \liminf_n C(p_n(x,y), q_n(x,y))\}.\end{aligned}$$

Note that by Fatou's Lemma,

$$\int_{X\times Y} C(p^*(x,y), q^*(x,y))\mathrm{d}\theta(x,y) \le \inf\ (4.6). \tag{4.11}$$

So if (p^*,q^*) was budget-constrained-feasible, the proof would be complete, but it is not necessarily the case that (p^*,q^*) is budget-constrained-incentive-compatible (and this is where assumption (4.8) comes into play). Note that by construction and using budget-constrained-feasibility of (p_n,q_n), we obviously have that for every $(x,y)\in X\times Y$, $p^*(x,y)\le y$, $V(x,p^*(x,y),q^*(x,y))\ge V(x,p_0,q_0)$; note also that $(p^*(x,y),q^*(x,y))\in\mathcal{A}^*$ because of (4.10), in particular since $p^*(x,y)\le y$ this gives

$$v^*(x,y) \ge V(x,p^*(x,y),q^*(x,y)),\ \forall (x,y)\in X\times Y. \tag{4.12}$$

From (4.2), (4.10), it is easy to deduce that

$$v^* \ge \limsup_n v_n. \tag{4.13}$$

Now observe that v^* is nondecreasing and upper semi-continuous with respect to its second argument. Hence, defining

$$v_-^*(x,y) := \lim_{\varepsilon\to 0^+} v^*(x, y-\varepsilon),\ \forall x\in X, \forall y\in Y\setminus\{\underline{y}\}, v_-^*(x,\underline{y}) := v^*(x,\underline{y}),$$

the (measurable) *singular* set

$$S := \{(x,y)\in X\times Y\ :\ v^*(x,y) > v_-^*(x,y)\}$$

has at most countable slices S_x for every $x\in X$. Assumption (4.8) thus implies that $\theta(S) = 0$. Note also that, again by (4.8), $\theta(X\times\{\underline{y}\}) = 0$. Therefore the *regular* set $R := (X\times(Y\setminus\{\underline{y}\}))\setminus S$ is of full measure for θ. Let now $(x,y)\in R$, $\varepsilon > 0$ be such that $y-\varepsilon\in Y$, by compactness of $\mathcal{A}^*$ and definition of v^* there is a $(p,q)\in\mathcal{A}^*$ such that $p\le y-\varepsilon$ and $v^*(x,y-\varepsilon) = V(x,p,q)$. Thanks to (4.10), there is a sequence (p_n,q_n) converging to (p,q) with $(p_n,q_n)\in\mathcal{A}_n$ for every n and $p_n\le y$ for large enough n so that

$$\liminf_n v_n(x,y) \ge \liminf_n V(x,p_n,q_n) = v^*(x,y-\varepsilon).$$

Letting $\varepsilon\to 0^+$ thus gives

$$\liminf_n v_n(x,y) \ge v_-^*(x,y). \tag{4.14}$$

Recalling (4.13) and using the fact that $v^* = v_-^*$ on R, we deduce that

$$v_n \to v^* \text{ on } R. \tag{4.15}$$

In particular if $(x,y)\in R$, since $v_n(x,y) = V(x,p_n(x,y),q_n(x,y))$ converges to $v^*(x,y)$, choosing a subsequence of $(p_n(x,y),q_n(x,y))$ converging to $(p^*(x,y),q^*(x,y))$ gives

$$v^*(x,y) = V(x,p^*(x,y),q^*(x,y)).$$

This enables us to conclude that for every $(x,y)\in R$, any $(x',y')\in X\times Y$, whenever $p^*(x',y')\le y$ one has $v^*(x,y) = V(x,p^*(x,y),q^*(x,y)) \ge V(x,p^*(x',y'),q^*(x',y'))$. The last step is to modify the contract (p^*,q^*) on a negligible set to make it budget-constrained feasible. To do this, first set

$$\widetilde{\mathcal{A}} := \{(p_0,q_0)\}\cup\overline{\{(p^*(x',y'),q^*(x',y')),\ (x',y')\in R\}}$$

and

$$\widetilde{v}(x,y) := \max_{(p,q)\in\widetilde{\mathcal{A}},\ p\le y} V(x,p,q),$$

and let $(\widetilde{p}, \widetilde{q})$ be a measurable selection of the set-valued map $(x, y) \mapsto \{(p, q) \in \widetilde{\mathcal{A}} : \ p \leq y, \ \widetilde{v}(x, y) = V(x, p, q)\}$. Since $\widetilde{v}(x, y) = v^*(x, y) = V(x, p^*(x, y), q^*(x, y))$ when $(x, y) \in R$, we may further impose that $(\widetilde{p}, \widetilde{q})$ and (p^*, q^*) coincide on R hence θ-almost everywhere. Then, it is straightforward to check that $(\widetilde{p}, \widetilde{q})$ is budget-constrained feasible, and the fact that it solves (4.6) directly follows from (4.11) and the fact that $(\widetilde{p}, \widetilde{q}) = (p^*, q^*)$ θ-almost everywhere.

□

Appendix: on measurable selections

We have invoked several times the existence of measurable selections of certain set-valued maps, we gather here some detailed justifications for the existence of such maps. Given a measurable space $(X, \mathcal{F})$, a Polish space Z and a set valued map $\Gamma : X \to 2^Z$ with nonempty values, a measurable selection of Γ is by definition an $\mathcal{F}$-measurable (single-valued) map z: $X \to Z$ such that $z(x) \in \Gamma(x)$ for all $x \in X$. A general existence result for measurable selections is given by the Kuratowski and Ryll-Nardzewski Theorem (see [7] and also the survey by Himmelberg [6]) which ensures that whenever

- $\Gamma(x)$ is closed and nonempty for every $x \in X$, and
- Γ is weakly-measurable in the sense that for every *open* subset U of Z, the set $\Gamma^{-1}(U) := \{x \in X \ : \ \Gamma(x) \cap U \neq \emptyset\}$ belongs to $\mathcal{F}$

then Γ admits a measurable selection.

In fact we do not use the full generality of the Kuratowski and Ryll-Nardzewski Theorem. The set-valued maps we have encountered through the paper satisfy a stronger measurability property than the one above, namely they satisfy that for every *closed* subset F of Z, $\Gamma^{-1}(F)$ belongs to $\mathcal{F}$ (to see that it implies weak-measurability it is enough to write the open set U as a countable union of closed sets). The first measurable selection result we have used is the following:

Lemma 4.4. *Let K be a nonempty compact subset of Z, z_n be a sequence of measurable maps, z_n: $X \to K$, and let C: $K \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous and not identically $+\infty$ on K. For all $x \in X$, let*

$$\Gamma(x) := \{z \in K \ : \ \exists n_j \to \infty \ : \ z_{n_j}(x) \to z, \ C(z_{n_j}(x)) \to \liminf_n C(z_n(x))\}.$$

Then Γ admits a measurable selection.

Proof. It is easy to check that $\Gamma(x)$ is a nonempty and closed subset of Z for every $x \in X$. As explained above, a sufficient condition for the existence of a measurable selection is that $\Gamma^{-1}(F)$ is measurable whenever F is closed but it is easy to check that

$$\Gamma^{-1}(F) = \{x \in X \ : \ \liminf_n (\mathrm{dist}(z_n(x), F) + C(z_n(x))) = \liminf_n C(z_n(x))\}$$

which written in this way is obviously measurable.

□

In the proof of Theorem 3.2, we have used:

Lemma 4.5. *Let U satisfy (2.1)-(2.2), A be a nonempty compact subset of Z and set*

$$v_A(x) := \max_{z \in A} U(x, z),$$

then v_A is measurable and setting for every $x \in X$,

$$\Gamma_A(x) := \{z \in A \ : \ U(x, z) = v_A(x)\}$$

then Γ_A admits a measurable selection.

Proof. The fact that v_A is measurable follows by taking $\{a_n\}_n$ a countable and dense subset of A and writing $v_A(x) = \lim_n \max_{k \leq n} U(x, a_k)$. Obvioulsy, $\Gamma_A(x)$ is nonempty and closed for every $x \in X$. Now if F is a closed subset of A we claim that $\Gamma_A^{-1}(F)$ is measurable, this set being empty when $A \cap F = \emptyset$ we may assume that the (compact) set $A \cap F \neq \emptyset$, then $\Gamma_A^{-1}(F)$ is the set where v_A and $v_{A \cap F}$ coincide, it is therefore measurable.

□

The following variant of Lemma 4.5 was used for the budget-constrained model:

Lemma 4.6. *Let V satisfy (4.2)-(4.3), A be a nonempty compact subset of $\mathbb{R} \times Q$ and set for every $(x, y) \in X \times Y$*

$$v_A(x, y) := \max_{(p,q) \in A, p \leq y} V(x, p, q),$$

(with the convention that $v_A(x, y) = -\infty$ whenever $A \cap (-\infty, y] \times Q = \emptyset$), then v_A is measurable and setting for every $(x, y) \in X \times Y$,

$$\Sigma_A(x, y) := \{(p, q) \in A \ : p \leq y, \ V(x, p, q) = v_A(x, y)\}$$

then Σ_A admits an $\mathcal{F} \otimes \mathcal{B}$-measurable selection.

Proof. For $\lambda > 0$ set

$$v_A^\lambda(x,y) := \max_{(p,q)\in A} \{V(x,p,q) - \lambda(p-y)_+)\}$$

thanks to Lemma 4.5, v_A^λ is measurable and it is easy to check that v_A^λ converges in a nonincreasing way to v_A as $\lambda \to +\infty$ which shows that v_A is measurable. The fact that Σ_A admits a measurable selection can then be shown as in the proof of Lemma 4.5.

□

Acknowledgements: Guillaume Carlier is grateful to the Agence Nationale de la Recherche for its support through the project MAGA (ANR-16-CE40-0014). Kelvin Shuangjian Zhang thanks the support of Mitacs Globalink Research Award during his visit to Guillaume Carlier at MOKAPLAN, INRIA-Paris in the summer 2018.

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